

POLYTOPES AND SKELETA

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ABSTRACT. To each simplicial reflexive polytope $\Delta \subset \mathbb{Z}^{n+1}$, we attach an n -dimensional space Λ^∞ . It is the Legendrian boundary of a conic Lagrangian considered in work of the authors, Fang and Liu [FLTZ1], and because of this it carries a sheaf of dg categories called the “Kashiwara-Schapira sheaf.” We discuss some conjectures and results about the role that Λ^∞ and the Kashiwara-Schapira sheaf should play in homological mirror symmetry.

1. INTRODUCTION

Locality in the Fukaya category of an exact symplectic manifold, and the relation of this structure to sheaf theory, has been studied in works of Abouzaid [A], Abouzaid-Seidel [AS], Kontsevich [K2], Seidel [S2], and, with Sibilla, the authors [STZ1]; and in unpublished works of Tamarkin, Nadler, and Nadler-Tanaka. A common thread is to identify a singular Lagrangian “skeleton,” and then relate the Fukaya category to the combinatorial structure of how this skeleton is built from smooth parts, as described by quivers or constructible sheaves. Typically, these skeleta are constructed from a choice of some additional structure placed on the exact symplectic manifold (e.g., Stein, Liouville, Weinstein, Morse), though the Fukaya category itself should be independent of such choices.

We wish to apply these ideas to Kontsevich’s homological mirror symmetry conjecture (HMS). In this note we describe an approach to the HMS for a certain class of toric hypersurfaces considered by Batyrev and Borisov. Locality, skeleta, and constructible sheaves appear at the large complex structure/large volume limits of these families of hypersurfaces.

Let Δ and Δ^\vee be a dual pair of reflexive polytopes. Batyrev and Borisov [Ba, Bo] explain how to construct from them a pair of smooth projective Calabi-Yau varieties Z and Z^\vee that should be mirror to each other.

Conjecture 1.1 (Kontsevich/Batyrev-Borisov HMS for toric hypersurfaces). *There is an equivalence of categories between vector bundles on Z and the Fukaya category on Z^\vee :*

$$\mathrm{Perf}(Z) \cong \mathrm{Fuk}(Z^\vee).$$

Let Z^∞ denote the “large complex structure limit” (LCL) of Z —it is a reducible and therefore singular complex algebraic variety with toric components. Let Z_∞^\vee denote the “large volume limit” (LVL) of Z^\vee —it is an affine hypersurface in an algebraic torus. We review the constructions of Z^∞ and Z_∞^\vee in Section 2. We have the following variant of Conjecture 1.1, proposed by Seidel [S1]:

Conjecture 1.2 (HMS at the limit). *There is an equivalence of categories*

$$\mathrm{Perf}(Z^\infty) \cong \mathrm{Fuk}_c(Z_\infty^\vee)$$

where the RHS denotes the Fukaya category of compact Lagrangian branes in Z_∞^\vee .

In this paper, in Sections 4 and 5, we define a topological space Λ^∞ which “could be” a Lagrangian skeleton of Z_∞^\vee . The space Λ^∞ is a Legendrian submanifold of a contact boundary of the cotangent bundle of a compact torus. The technology of Kashiwara and Schapira, reviewed in Section 3, allows us to equip Λ^∞ with a sheaf of dg categories.

We conjecture that this sheaf of categories is equivalent to the sheaf of Fukaya categories on the skeleton of Z_∞^\vee . In forthcoming work with Sibilla [STZ2], we will prove that the category of global objects of this sheaf is equivalent to the category of perfect complexes on Z^∞ . If our conjecture is accurate, this would establish HMS at the LCL/LVL limits for a large class of toric hypersurfaces.

A subconjecture is that Λ^∞ is homotopy equivalent to the hypersurface Z_∞^\vee . As evidence for this, we study the Leray spectral sequence of a map $\Lambda^\infty \rightarrow \partial\Delta^\vee$, which we conjecture degenerates at E^2 over the rational numbers (this is not always true over the integers). We have calculated the E^2 page in many examples by computer – see Section 6. Our results agree with the Betti numbers computed by Danilov-Khovanskii [DK].

Acknowledgments. We thank Bohan Fang, Chiu-Chu Melissa Liu, and Nicolò Sibilla for many conversations and for collaborating on related work. We thank David Nadler and Dima Tamarkin for their insights. The work of EZ is supported by NSF-DMS-1104779.

2. REFLEXIVE POLYTOPES AND MIRROR SYMMETRY

Let M be a lattice, and let M^\vee denote the dual lattice. A pair of lattice polytopes $\Delta \subset M$ and $\Delta^\vee \subset M^\vee$ are called a *reflexive pair* if they are polar duals of each other:

$$\begin{aligned}\Delta^\vee &= \{m^\vee \in M^\vee \mid \forall m \in \Delta \quad \langle m^\vee, m \rangle \geq -1\} \\ \Delta &= \{m \in M \mid \forall m^\vee \in \Delta^\vee \quad \langle m^\vee, m \rangle \geq -1\}\end{aligned}$$

To a reflexive pair of lattice polytopes we associate a pair of toric varieties X and X^\vee . Let us make the following toric notation:

- We set $N = M^\vee$ and $N^\vee = M$. Our hope is that, for those used to “Fulton’s notation” [Fu] in toric geometry, having four names for these two lattices will *reduce* confusion.
- Write $N_\mathbb{R} = M_\mathbb{R}^\vee$ and $N_\mathbb{R}^\vee = M_\mathbb{R}$ for the realification of these lattices. We will often not distinguish $\Delta \subset M$ (resp. $\Delta^\vee \subset M^\vee$) from its convex hull in $M_\mathbb{R}$ (resp. $M_\mathbb{R}^\vee$).
- Let $\mathbf{T} = N \otimes \mathbb{C}^*$ be the algebraic torus whose character lattice is M , and let $\mathbf{T}^\vee = N^\vee \otimes \mathbb{C}^*$ be the algebraic torus whose character lattice is M^\vee .

We may associate a pair of toric varieties X and X^\vee to a reflexive pair of lattice polytopes. X is a compactification of \mathbf{T} and X^\vee is a compactification of \mathbf{T}^\vee , and they are characterized by the following properties: X is Gorenstein, its anticanonical bundle is ample, and the moment polytope of the anticanonical bundle is Δ .

Following Batyrev and Borisov [Ba, Bo], we expect a mirror relationship between a smooth projective anticanonical hypersurface $Z \subset X$ and a smooth projective anticanonical hypersurface $Z^\vee \subset X^\vee$. Our purpose is to understand this mirror relationship at the large complex structure/large volume limits.

Remark 2.1. From now on, we only study:

- *complex* geometry of X , its hypersurfaces and other related spaces
- *symplectic* geometry of X^\vee , its hypersurfaces and other related spaces

That is, we consider just one direction of mirror symmetry at a time.

Actually it is now necessary to make some corrections, as in Batyrev-Borisov's recipe the singularities of the pairs (X, Z) and (X^\vee, Z^\vee) might need to be resolved. We will restrict our attention to pairs of reflexive polytopes where a resolution process is unnecessary: from now on we assume that Δ is vertex-simplicial, and Δ^\vee is facet-simplicial. In that case it is natural to regard X as a smooth Deligne-Mumford stack instead of a variety with Gorenstein singularities. (We explain how to do this in detail in Sections 4 and 5.) The toric variety X^\vee could still be quite singular, but these singularities are not visible to the hypersurface at the large volume limit.

- (1) The *large complex structure limit* or LCL is the union of the toric divisors of X , regarded as a Deligne-Mumford stack. We denote it by Z^∞ . As Δ is vertex-simplicial, it has toric components with normal crossings singularities.
- (2) We may write any polynomial function $f : \mathbf{T}^\vee \rightarrow \mathbb{C}$ as a finite linear combination of characters. The *Newton polytope* of f is the convex hull of the characters that appear; it is a lattice polytope in M^\vee . The *large volume limit* or LVL is any generic affine hypersurface Z_∞^\vee in \mathbf{T}^\vee whose defining equation has Newton polytope Δ^\vee .

Z^∞ evidently inherits a complex structure (necessarily singular, and possibly with some stackiness) from X . We endow Z_∞^\vee with a symplectic structure (necessarily exact) induced from $\sum \frac{1}{r} dr_i \wedge d\theta_i$, where r_i, θ_i are polar coordinates on \mathbf{T}^\vee .

Remark 2.2. Though Z^∞ is canonically defined, Z_∞^\vee depends on a choice of Laurent polynomial $f : \mathbf{T}^\vee \rightarrow \mathbb{C}$.

With this notation, we can expect Conjecture 1.2 to hold under a simplicial assumption:

Conjecture 2.3 (HMS). *Suppose Δ is vertex-simplicial and Δ^\vee is facet-simplicial. Then there is an equivalence*

$$\mathcal{P}\text{erf}(Z^\infty) \cong \text{Fuk}_c(Z_\infty^\vee)$$

where the right-hand side denotes the Fukaya category of compact Lagrangian branes in Z_∞^\vee .

3. MICROLOCAL SHEAF THEORY

In this section we recall some notions from the microlocal sheaf theory of Kashiwara-Schapira [KS], and its relationship to Fukaya theory [NZ, N]. In Section 3.1, we introduce the “Kashiwara-Schapira sheaf” of dg categories, which is a variant of the theory of [KS, Chapter VI]. In Section 3.2 we discuss how this sheaf behaves under “conormal extension.”

If Y is a real analytic manifold, we let $Sh(Y)$ denote the dg triangulated category of sheaves on Y that are constructible with respect to an arbitrary subanalytic Whitney stratification of Y . To each object F of $Sh(Y)$ we can attach a conic Lagrangian subset $SS(F) \subset T^*Y$, called the singular support of F . Given a conic subset $\Lambda \subset T^*Y$ the full subcategory of $Sh(Y)$ spanned by objects F with $SS(F) \subset \Lambda$ is triangulated, we denote it by $Sh(Y; \Lambda)$.

Now $Sh(Y; \Lambda)$ is equivalent to $\text{Fuk}(T^*Y; \Lambda)$, the Fukaya category generated by exact Lagrangian branes L with $L^\infty \subset \Lambda^\infty$ at contact infinity [NZ, N]. It is widely expected that $\text{Fuk}(T^*Y; \Lambda)$ represents the global sections of a *sheaf* of Fukaya categories on T^*Y supported on Λ .

3.1. The Kashiwara-Schapira sheaf of categories on conic Lagrangians and on Legendrians. Fix a manifold Y and a conic Lagrangian Λ , so that $\text{Fuk}(T^*Y; \Lambda)$ can be computed in terms of constructible sheaves, as above. Microlocal sheaf theory is well-suited to studying the expected sheaf of Fukaya categories on Λ : using constructible sheaves, we can define a sheaf of categories with no Fukaya theory in place.

If $\Omega \subset T^*Y$ is a conic open subset, Kashiwara and Schapira define the category of “microlocal sheaves on Ω ” to be the quotient $Sh(Y)/Sh(Y; T^*Y - \Omega)$. Let us denote this category by $MSh^p(\Omega)$. We may regard $\Omega \mapsto MSh^p(\Omega)$ as a contravariant functor from the poset of conic open subsets of T^*Y to the ∞ -category of dg triangulated categories. In other words, MSh^p is a presheaf of dg categories on T^*Y in its conic topology.

For $\Lambda \subset T^*Y$ a conic Lagrangian, we can define a smaller variant of MSh^p we call MSh_Λ^p :

$$MSh_\Lambda^p(\Omega) = Sh(Y; \Lambda \cup (T^*Y - \Omega)) / Sh(Y; T^*Y - \Omega)$$

This again is a presheaf of dg categories on T^*Y whose value on any open set that does not meet Λ is zero. It follows that the sheafification of MSh_Λ^p is supported on Λ , and we can regard it as a sheaf of dg categories on Λ itself in its conic topology. We denote this sheaf of dg categories by MSh_Λ .

Remark 3.1. Of course MSh_Λ^p and MSh_Λ give different categories over open subsets of T^*Y , but it is not difficult to show that the sheafification process does not change *global* sections: $MSh_\Lambda(T^*Y) \cong MSh_\Lambda^p(T^*Y) := Sh(Y; \Lambda)$.

Remark 3.2. If Λ is a conic Lagrangian let us write Λ^∞ for the associated Legendrian subset of the contact boundary of T^*Y . That is, $\Lambda^\infty = (\Lambda - (\Lambda \cap Y)) / \mathbb{R}_{>0}$. As MSh_Λ is a sheaf in the conic topology on Λ it induces a sheaf on Λ^∞ , which we will denote by MSh_{Λ^∞} .

We expect that the Kashiwara-Schapira sheaf MSh_{Λ^∞} agrees with the anticipated Fukaya sheaf of categories, though to our knowledge the latter object has not yet been constructed.

3.2. Conormal extension. Let Y be a manifold and let $i : Y' \hookrightarrow Y$ be a closed submanifold. We have a short exact sequence of vector bundles on Y'

$$0 \rightarrow T_{Y'}^*Y \rightarrow T^*Y|_{Y'} \xrightarrow{\pi} T^*Y' \rightarrow 0$$

We may regard the middle term of this sequence as a closed subset of T^*Y . If Λ is a conic Lagrangian in T^*Y' , then $\pi^{-1}(\Lambda)$ is a conic Lagrangian of T^*Y that we call the *full conormal extension* of Λ .

The functor $i_* : Sh(Y') \rightarrow Sh(Y)$ carries $Sh(Y'; \Lambda) \rightarrow Sh(Y; \pi^{-1}(\Lambda))$ —in fact this latter functor is an equivalence. It turns out that this functor induces an equivalence between the Kashiwara-Schapira sheaves on Λ and on $\pi^{-1}(\Lambda)$.

Proposition 3.3 ([STZ2]). *The extension-by-zero functor i_* induces an equivalence*

$$\pi^{-1}MSh_\Lambda \rightarrow MSh_{\pi^{-1}(\Lambda)}$$

A basic consequence is the following. If σ° is a bundle of open convex cones of the conormal bundle of Y' in Y , then $\sigma^\circ \cap \pi^{-1}(\Lambda)$ is an open subset of $\pi^{-1}(\Lambda)$, which we call the *conormal extension of Λ determined by σ°* . The Proposition and basic properties of pullback sheaves give us an identification

$$MSh_{\pi^{-1}(\Lambda)}(\Omega) \cong Sh(Y'; \Lambda)$$

whenever Ω is a conormal extension of Λ determined by an open convex cone.

4. SKELETA FROM TORIC VARIETIES

We wish to probe the HMS conjecture 2.3 using the coherent-constructible correspondence of [FLTZ1] and [FLTZ2]. We will apply the theory of those papers to the toric orbifold X of Section 2 and its subvarieties, but let us start by recalling some features of the general case.

Let us first assume that \mathcal{X} is a toric variety with no stackiness. Let $\Sigma \subset \mathcal{N}_{\mathbb{R}}$ be the fan associated to \mathcal{X} , where $\mathcal{N}_{\mathbb{R}} = \mathcal{N} \otimes \mathbb{R}$ for a suitable lattice \mathcal{N} . Then, letting $\mathcal{M} = \text{Hom}(\mathcal{N}, \mathbb{Z})$ and $\mathcal{M}_{\mathbb{R}} = \text{Hom}(\mathcal{N}_{\mathbb{R}}, \mathbb{R})$, we (with Fang and Liu) defined in [FLTZ1] a conic Lagrangian subset of $T^*(\mathcal{M}_{\mathbb{R}}/\mathcal{M})$, as follows:

$$\Lambda_{\Sigma} = \bigcup_{\sigma \in \Sigma} (\sigma^{\perp} + \mathcal{M}) \times -\sigma$$

Here $\sigma^{\perp} \subset \mathcal{M}_{\mathbb{R}}$ denotes the collection of linear covectors on $\mathcal{N}_{\mathbb{R}}$ that vanish on every element of $\sigma \in \Sigma$, and $\sigma^{\perp} + \mathcal{M}$ denotes its image in $\mathcal{M}_{\mathbb{R}}/\mathcal{M}$. As σ is rational, $\sigma^{\perp} + \mathcal{M}$ is a subtorus of $\mathcal{M}_{\mathbb{R}}/\mathcal{M}$, and $(\sigma^{\perp} + \mathcal{M}) \times -\sigma$ can be identified with an open subset of its conormal bundle.

When \mathcal{X} is an orbifold we have the following variant from [FLTZ2]. Actually we will give a slightly more general treatment here than in [FLTZ2], and allow \mathcal{X} to have some “gerbiness” or generic isotropy. From \mathcal{X} we get a stacky fan in the sense of [BCS], which we will denote by Σ . In more detail, Σ consists of the following data:

- A finitely generated abelian group \mathcal{N} . (In our applications, \mathcal{N} will be a quotient of the group N of Section 2.)
- A complete simplicial fan $\Sigma \subset \mathcal{N}_{\mathbb{R}} := \mathcal{N} \otimes_{\mathbb{Z}} \mathbb{R}$
- A collection $\beta = \{\beta_i\}$, one β_i for each ray (i.e., one-dimensional cone) $\rho_i \in \Sigma$, such that $\beta_i \in \mathcal{N}$ maps to a nonzero element $\bar{\beta}_i$ of ρ_i under the natural map $\mathcal{N} \rightarrow \mathcal{N}_{\mathbb{R}}$.

To define the conic Lagrangian associated to a stacky fan [FLTZ2, Definition 6.3], we replace the ambient torus $\mathcal{M}_{\mathbb{R}}/\mathcal{M}$ by the Pontrjagin dual of \mathcal{N} , i.e. by the group G of homomorphisms $\mathcal{N} \rightarrow \mathbb{R}/\mathbb{Z}$. If \mathcal{N} has torsion, then G may be disconnected. We replace the subtori $\sigma^{\perp} + \mathcal{M}_{\mathbb{Z}}$ of formula (4) by closed subgroups (again, possibly disconnected) $G_{\sigma} \subset G$. Specifically, for each $\sigma \in \Sigma$, G_{σ} is the group of homomorphisms $\phi : \mathcal{N} \rightarrow \mathbb{R}/\mathbb{Z}$ that have $\phi(\beta_i) = 1$ whenever $\beta_i \in \sigma$. Then we define Λ_{Σ} by

$$\Lambda_{\Sigma} = \bigcup_{\sigma \in \Sigma} G_{\sigma} \times -\sigma$$

The results from [FLTZ1, FLTZ2] give us the following:

Theorem 4.1 (Coherent-constructible correspondence, or “CCC”). *There is a full embedding*

$$\kappa : \text{Perf}(\mathcal{X}) \hookrightarrow \text{Sh}(G; \Lambda_{\Sigma})$$

Conjecturally, this embedding is an equivalence—the “equivariant” version of this is proven in [FLTZ1, FLTZ2].

4.1. Understanding the Kashiwara-Schapira sheaf on Λ_{Σ} . The sheaf structure of $M\text{Sh}_{\Lambda_{\Sigma}}$ gives us a restriction operation

$$\text{Sh}(G; \Lambda_{\Sigma}) \rightarrow M\text{Sh}_{\Lambda_{\Sigma}}(\Omega)$$

for each open subset $\Omega \subset \Lambda_{\Sigma}$. For certain Ω these operations have coherent counterparts in the CCC. We describe these in this section.

If \mathcal{X} is a toric variety with fan $\Sigma \subset \mathcal{N}_{\mathbb{R}}$, then the closure of any torus orbit is also a toric variety in a natural way. The structure of this toric subvariety is visible in the combinatorics of the fan. Each cone $\sigma \in \Sigma \subset \mathcal{N}_{\mathbb{R}}$ has an associated “normal fan”

$$\Sigma(\sigma) \subset \mathcal{N}_{\mathbb{R}}/(\mathbb{R} \cdot \sigma)$$

consisting of the projections of those $\tau \in \Sigma$ with $\tau \supset \sigma$.

These remarks extend to stacky fans. If \mathcal{N} is a finitely generated abelian group and $\Sigma = (\Sigma \subset \mathcal{N}_{\mathbb{R}}, \beta \subset \mathcal{N})$ is a stacky fan, then for each σ we may form the quotient $\mathcal{N}(\sigma) = \mathcal{N}/(\mathbb{Z} \cdot \{\beta_i : \bar{\beta}_i \in \sigma\})$. Then the fan $\Sigma(\sigma)$ lives naturally in $\mathcal{N}(\sigma)_{\mathbb{R}}$, and the subcollection $\beta_{\sigma} = \{\beta_j : \bar{\beta}_j \in \tau \setminus \sigma \text{ for some } \tau \supset \sigma\}$ of β_j with $\bar{\beta}_j$ in a cone containing σ but not in σ itself, forms a stacky structure on this fan. We denote the resulting stacky fan by $\Sigma(\sigma)$. There is a natural inclusion of toric stacks $\mathcal{X}_{\Sigma(\sigma)} \hookrightarrow \mathcal{X}_{\Sigma}$.

The skeleton $\Lambda_{\Sigma(\sigma)}$ is a conic Lagrangian in the cotangent bundle of the subgroup $G_{\sigma} \subset G$. We define an open subset of $\Lambda_{\Sigma(\sigma)} \subset \Lambda_{\Sigma}$ as follows:

$$\Lambda_{\Sigma}(\sigma) = \bigcup_{\tau \supset \sigma} G_{\tau} \times (-\tau^{\circ})$$

This is just the conormal extension of $\Lambda_{\Sigma(\sigma)}$ by the trivial bundle of cones determined by σ° (see Section 3.2).

Theorem 4.2. *We have a commutative square*

$$\begin{array}{ccc} \mathrm{Perf}(\mathcal{X}_{\Sigma}) & \xrightarrow{\quad\quad\quad} & \mathrm{Perf}(\mathcal{X}_{\Sigma(\sigma)}) \\ \kappa \downarrow & & \downarrow \kappa \\ MSh_{\Lambda_{\Sigma}}(\Lambda_{\Sigma}) & \xrightarrow{\quad res \quad} MSh_{\Lambda_{\Sigma}}(\Lambda_{\Sigma}(\sigma)) \xrightarrow{\quad \cong \quad} MSh_{\Lambda_{\Sigma(\sigma)}}(\Lambda_{\Sigma(\sigma)}) \end{array}$$

The top row is given by restriction of vector bundles, the columns are given by the CCC, and the bottom row is given by restriction in the Kashiwara-Schapira sheaf and conormal extension.

5. SKELETA FROM TORIC HYPERSURFACES

Now we return to the Batyrev-Borisov hypersurfaces. Let $\Delta, \Delta^{\vee}, X, X^{\vee}$ and $Z^{\infty}, Z_{\infty}^{\vee}$ be as in Section 2. We assume that Δ is vertex-simplicial, and Δ^{\vee} is facet-simplicial, so that in particular we regard X as a smooth Deligne-Mumford stack. In more detail, X is determined by a stacky fan $\Sigma = (\Sigma, \{\beta_i\})$ that is closely related to $\Delta^{\vee} \subset M^{\vee} = N$. In the notation of Section 4, we have $\mathcal{N} = N$, a free abelian group. The cones of Σ are indexed by the faces of Δ^{\vee} , by setting $\sigma_F := \mathbb{R}_{\geq 0} \cdot F \subset N_{\mathbb{R}}$, and the β_i are precisely the vertices of Δ^{\vee} .

Now we attach to Σ the conic Lagrangian $\Lambda = \Lambda_{\Sigma}$ as in Section 4. As $M_{\mathbb{R}} = N_{\mathbb{R}}^{\vee}$ and $M = N^{\vee}$, after choosing an inner product we may identify $T^*(M_{\mathbb{R}}/M)$ with the complex algebraic torus \mathbf{T}^{\vee} . The CCC and microlocalization suggest that $\Lambda \subset \mathbf{T}^{\vee}$ is the skeleton of a Weinstein structure¹ on \mathbf{T}^{\vee} that is suited for studying the mirror of X . We propose that the contact boundary of Λ , i.e. $\Lambda^{\infty} \subset M_{\mathbb{R}}/M \times (N_{\mathbb{R}} - 0)/(\mathbb{R}_{>0})$, is suited for studying the mirror Z_{∞}^{\vee} of Z^{∞} . In particular Λ^{∞} should be homeomorphic to a skeleton of Z_{∞}^{\vee} .

¹A Weinstein manifold is a complete, exact symplectic manifold whose Liouville vector field is gradient-like for an exhausting Morse function. The skeleton is the union of stable manifolds of Liouville flow.

Conjecture 5.1. *Suppose Δ is vertex simplicial and Δ^\vee is facet simplicial. There is a Weinstein structure on Z_∞^\vee whose Lagrangian skeleton is homeomorphic to Λ^∞ . The sheaf of Fukaya categories on this skeleton is equivalent to the Kashiwara-Schapira sheaf on Λ^∞ .*

In view of mirror symmetry, the following result is strong evidence for the Conjecture.

Theorem 5.2. *The category of global sections of the Kashiwara-Schapira sheaf on Λ^∞ contains $\text{Perf}(Z^\infty)$ as a full subcategory.*

A proof will appear in forthcoming work with Sibilla [STZ2]. As discussed in Section 4.1, the normal stacky fans $\Sigma(\sigma)$ determine an open cover $\{\Lambda_{\Sigma}(\sigma)\}$ of Λ_{Σ} . If we omit the zero cone, we get a closely related open cover $\{\Lambda_{\Sigma}(\sigma)/\mathbb{R}_{>0}\}$ cover Λ^∞ which can be used to compute the global sections of the Kashiwara-Schapira sheaf. Theorems 4.2 yields the full embedding Theorem 5.2 by applying the CCC to each component of Z^∞ and each open chart of Λ^∞ . This full embedding is an equivalence as long as the conjecture alluded to after Theorem 4.1 holds.

Concrete Description of Λ^∞ . In the notation of Section 4, we begin with $\mathcal{N} = N$, a free abelian group. Each face F of Δ^\vee labels a cone $\sigma_F := \mathbb{R}_{\geq 0} \cdot F \subset N_{\mathbb{R}}$. The stacky fan $\Sigma(\sigma_F)$ has associated abelian group $N_F := \mathcal{N}(\sigma_F) := N/(\mathbb{Z} \cdot V_F)$, not necessarily free; here V_F is the set of vertices of F . Then $M_F = V_F^\perp$. Cones of $\Sigma(\sigma_F)$ are labeled by faces $F' \supset F$ and defined by $\sigma_{F'}^F = \mathbb{R}_{\geq 0} \cdot V_{F'}$, or rather the image of this set in $\mathcal{N}(\sigma_F)_{\mathbb{R}} = N_{\mathbb{R}}/(\mathbb{R} \cdot V_F)$. The set of lifts of ray vectors is $\beta_F = \bigcup_{F' \supset F} V_{F'} \setminus V_F$. The associated abelian group is $G_{F'} := G_{\sigma_{F'}} = \{m \in M_{\mathbb{R}} : m(V_{F'}) \subset \mathbb{Z}\}/M \subset M_{\mathbb{R}}/M$. Note that this group depends only on F' , not the pair (F', F) . We have a short exact sequence of abelian groups $1 \rightarrow G_{F'}^\circ \rightarrow G_{F'} \rightarrow P_{F'} \rightarrow 1$, where $G_{F'}^\circ := M_{F', \mathbb{R}}/M = V_{F'}^\perp/M$ is the neutral component of $G_{F'}$, a torus, and $P_{F'} = \pi_0(G_{F'})$ is a finite quotient. Then $\Lambda' := \bigcup_F G_F \times F^\circ$ is the union of conormal extensions of the skeleta for the toric components which comprise Z^∞ . We define $\Lambda^\infty = \Lambda'/\mathbb{R}_{>0}$, which has an equivalent microlocal sheaf.

6. THE HOMOLOGY OF Λ^∞

A consequence of Conjecture 5.1 is that we have a homotopy equivalence between Z_∞^\vee and Λ^∞ . A weaker conjecture then states that these spaces have the same rational homology. We will discuss a computer program that makes many tests of this weaker conjecture in Section 6.2.

The space Λ^∞ projects naturally onto $(N_{\mathbb{R}} - 0)/\mathbb{R}_{>0}$, which we can identify with $\partial\Delta^\vee$. This induces a Leray spectral sequence abutting to the homology of Λ^∞ . The projection is trivial over the interior of each face of $\partial\Delta^\vee$, so that

$$\Lambda^\infty = \bigcup_F G_F \times F^\circ$$

where $G_F := G_{\sigma_F}$ are the disconnected tori defined in Section 4 and the union is over proper faces of Δ^\vee —i.e. G_F is the group of maps from N to \mathbb{R}/\mathbb{Z} that carries each $\beta_i \in F$ to 1.

The natural inclusion $\phi_{F'}^F : G_{F'} \hookrightarrow G_F$ for $F \subset F'$ describes the attaching of $G_F \times F^\circ$ to $G_{F'} \times F'^\circ$. The E^1 page of the Leray spectral sequence is the bi-graded chain complex

$$0 \leftarrow \bigoplus_{\dim F=0} H_*(G_F) \leftarrow \bigoplus_{\dim F=1} H_*(G_F) \leftarrow \bigoplus_{\dim F=2} H_*(G_F) \leftarrow \cdots$$

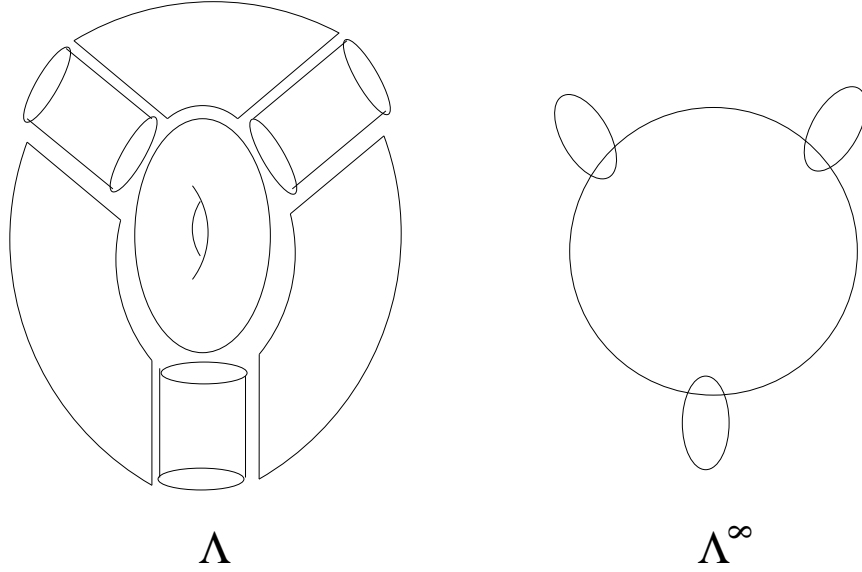


FIGURE 1. On the left is a rough picture of how the conic Lagrangian Λ attached to \mathbb{P}^2 should be constructed from constituents $G_\sigma \times -\sigma$. On the right is the Legendrian Λ^∞ that we expect to be a skeleton of $Z_\infty^\vee = \{(x, y) \in \mathbb{C}^* \times \mathbb{C}^* \mid ax + by + c(xy)^{-1} + d = 0\}$.

whose differentials are alternating sums of the maps induced by $\phi_{F'}^F$. We conjecture that with rational coefficients the spectral sequence degenerates, i.e. $E^2 = E^\infty$.

Explicitly, we define a chain complex (C_*, ∂) as follows. For each face F , let $P_F = \pi_0(G_F)$ and let $G_F^\circ \subset G_F$ be the neutral component of G_F , so that we have a short exact sequence

$$1 \rightarrow G_F^\circ \rightarrow G_F \rightarrow P_F \rightarrow 1,$$

with $G_F^\circ = M_{F, \mathbb{R}}/M_F$ (or $M_F = H_1(G_F^\circ)$). We have canonically $H_*(G_F) = (\bigwedge^* M_F)^{\oplus |P_F|}$. Pick an orientation of each face of Δ^\vee , so that for each pair of faces F', F with F of codimension one in F' we get a sign $\epsilon_{F, F'} = \pm 1$. Then for each $\gamma \in (\bigwedge^r M_F)^{\oplus |P_F|}$ we set

$$\partial\gamma = \sum_{F'} \epsilon_{F, F'} (\phi_{F'}^F)_*(\gamma)$$

If $\partial\gamma = 0$ then γ contributes to $H_{\dim(F)+r}(\Lambda^\infty)$.

6.1. Example. Consider the triangle $\Delta^\vee = \text{conv}(\{(2, -1), (-1, 2), (-1, -1)\}) \subset \mathbb{R}^2 = M^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, where $M^\vee = \mathbb{Z}^2$. Let $F = \{(-1, -1)\}$ so $\sigma_F = \mathbb{R}_{\geq 0} \cdot \{(-1, -1)\}$ and $\mathcal{N}(\sigma_F) = \mathbb{Z}^2 / \{(-1, -1)\} \cong \mathbb{Z}$, with $(a, b) \sim (a - b) \in \mathbb{Z}$. Then if $F' \supset F$ is the edge containing $(2, -1)$ and $(-1, -1)$, $\sigma_{F'}^F$ is the $\mathbb{R}_{\geq 0}$ span of $(3, 0) \sim 3 \in \mathbb{Z}$. Let $F'' \supset F$ be the remaining edge. Then $\sigma_{F''}^F$ is the span of $-3 \in \mathbb{Z}$. Further, $\beta_F = \{3, -3\}$ and this gives the stacky fan structure for the toric stack $\mathbb{P}^1(3, 3)$, a \mathbb{P}^1 with $\mathbb{Z}/3\mathbb{Z}$ orbifold points at the two poles. $G_F = \{(a, b) : -a - b \in \mathbb{Z}\} / \mathbb{Z}^2 \cong S^1$, so $H_*(G_F) \cong \mathbb{Z} \oplus \mathbb{Z}[-1]$. A similar story holds at each vertex. For the edge F' we have $G_{F'} = \{(a, b) : -a - b \in \mathbb{Z}, 2a - b \in \mathbb{Z}\} / \mathbb{Z}^2 \cong \mathbb{Z}/3\mathbb{Z}$. We can label the three points $(k/3, 1 - k/3)$,

and note that under $G_{F'} \hookrightarrow G_F$ they are distributed evenly on S^1 , and this explains how to attach chains to make Λ^∞ . As chains in Λ^∞ , these three points contribute in dimension $0+1=1$, since they lie over edges. A similar story holds at each edge.

The space of chains looks like \mathbb{Q}^3 in degree zero (coming from the vertices) and $\mathbb{Q}^3 \oplus \mathbb{Q}^9$ in degree one (the first factor coming from the vertices, the second from the edges). The only nonzero part of the differential is its restriction to the edges: $\mathbb{Q}^9 \rightarrow \mathbb{Q}^3$, and its form comes purely from the combinatorics of the polygon Δ^\vee , repeated three times because of the $\mathbb{Z}/3\mathbb{Z}$'s. The 3×9 matrix representing this map is

$$\begin{pmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We conclude that the image is two-dimensional, and hence $\dim H_0(C_*) = 3 - 2 = 1$, i.e. $H_0(C_*) = H_0(\Lambda^\infty) \cong \mathbb{Q}$ while $H_1(C_*) = H_1(\Lambda^\infty) \cong \mathbb{Q}^3 \oplus \mathbb{Q}^7$.

The polygon Δ^\vee defines the line bundle $\mathcal{O}_{\mathbb{P}^2}(3)$, and the affine cubic is an elliptic curve minus nine points at infinity (three for each toric divisor). It is connected with Euler characteristic $0 - 9$, hence $\dim H_0(Z_\infty^\vee) = 1$ and $\dim H_1(Z_\infty^\vee) = 10$, which agrees with our computation.

6.2. Computer program. It is straightforward to automate the process of calculating the homology of (C_*, ∂) for any polytope, including for example all facet-simplicial reflexive polytopes in dimension three. We have done this and posted the Sage worksheet in a public folder at <http://dl.dropbox.com/u/24939613/skeleton-homology.sws> (entering the URL in a browser will trigger a download of the file). Here we explain the algorithm. The program code and the example cell are also generously annotated.

The program runs by calculating the ranks and kernels of matrices encoding the maps $(\varphi_{F'}^F)_*$ on the \mathbb{Z} -modules $H_*(G_F, \mathbb{Z})$. Since $H_*(G_F) \cong (\bigwedge^* M_F)^{\oplus |P_F|}$, after we choose a \mathbb{Z} -basis e_i , $i \in \{1, \dots, s\}$, for M_F this determines a basis e_I^k for $\bigwedge^* M_F$, one basis element for each subset $I \subset \{1, \dots, s\}$ and element $k \in P_F$, or $2^s |P_F|$ in all.

First choose an identification $N \cong \mathbb{Z}^n$ and suppose $\Delta^\vee = \text{conv}\{v_1, \dots, v_p\}$, with $v_i = (v_{i,1}, \dots, v_{i,n})$. A face F is indexed by a subset $J = \{j_1 < j_2 < \dots\} \subset \{1, \dots, p\}$ indexing its vertices. Let A_J be the matrix with $(A_J)_{a,b} = v_{j_a,b}$. By changing bases for the lattice $\mathbb{Z} \cdot V_F$ and for N , we can perform row *and* column operations to put A_J in Smith normal form (computed by `smith_form()` in Sage)

$$S_J = U_J A_J V_J,$$

where S_J is diagonal with entries s_{11}, \dots, s_{rr} . These matrices fix an identification of G_F with $(\mathbb{R}/\mathbb{Z})^s \times \mathbb{Z}/s_{11} \times \dots \times \mathbb{Z}/s_{rr}$.

If F is a face corresponding to $J \subset \{1, \dots, p\}$ and $F' \supset F$ is a face of one dimension higher corresponding to $K \supset J$, the matrix $V_J^{-1} V_K$ encodes the information of the map $\phi_{F'}^F$.

The southeastern $(s+1) \times s$ block represents a linear map from $M_{F'}$ to M_F . The linear maps from $\bigwedge^* M_{F'}$ to $\bigwedge^* M_F$ are described by all the various minors of this block. The northwestern $(r-1) \times r$ block of $V_J^{-1} V_K$ encodes the map of discrete groups: to find out where the element $(k_1, \dots, k_r) \in P_{F'}$ goes, act on it by the matrix and read off the answer as an element of the group P_F . (Note that in restricting to these blocks, we are ignoring some information, *irrelevant to homology*, in the matrix $V_J^{-1} V_K$ which describes where the discrete group $P_{F'}$ (in the induced splitting) maps into the group G_F .)

With these data in hand, the only other information needed to implement the homology calculation is the differential of the polytope, and that is accomplished through simple combinatorics.

6.3. Comparisons. Danilov and Khovanskii [DK] have computed the ranks of the homology groups (or rather the dual compactly supported cohomology) of affine hypersurfaces in toric varieties. We have compared dimensions and verified that our chain complex computes the right betti numbers, as conjectured, for all 194 facet-simplicial three-dimensional reflexive polytopes.

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